A quadrature approach to the generalized frictionless shearing contact problem

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Abstract

In this study, the generalization of a frictionless contact problem in case of shearing deformation for an elastic inhomogeneous half space is presented. The basic equations of the elasticity theory and Fourier transform technique are applied to the problem to derive the system of singular integral equations. The obtained system of singular integral equations is solved by a quadrature approach. The numerical results are presented for the case of $N = 1$, $N = 2$, $N = 3$, where N denotes the number of the punches whose base are flat.

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1 Introduction

In literature, there are many studies of contact problems for an elastic layer. Some of them are listed below:

Contact problems for inhomogeneous layers in the view of elasticity theory are presented in [1, 2]. Generalova and Kovalenko examined the effect of a strip shaped punch on a linearly deformable foundation in [3]. They used the Chebyshev and Legendre polynomials and quadrature approximation to obtain the numerical results. Sing et.al. handle the contact problem in which the nonhomogeneous medium is bounded to another nonhomogeneous medium in [4]. The problem is reduced to dual integral equations by using Fourier cosine transforms. The numerical results are obtained by solving the Fredholm integral equations which are obtained from dual integral equations. Kahya et.al. study frictionless contact problem for a two-layer orthotropic elastic medium loaded through a rigid flat stamp in [5].

In the present paper, the problem given in [1, 6] is generalized for the case of N-punches. Also, by using a theorem, the obtained system of integral equation is converted to the system of singular integral equations. The numerical results are obtained in the view of the index theory given in [7, 8].

2 Statement of the problem

In this section, the contact problem is defined as in Figure 1. It is assumed the problem is perfectly bonded to both the coating width H and half-plane substrate. The punches are in contact with the elastic inhomogeneous coating along $y = 0$, $x \in \bigcup_{i=1}^{N}(a_i, b_i)$, where $(a_i, b_i) \cap (a_j, bj) = \varnothing, i, j = 1, 2, ..., N$ and subjected to forces P_i, $i = 1, 2, ..., N$, which are parallel to axis z and outside of the punches the surface is traction-free. The $i - th$ punch displaces as ε_i by the effect of the forces Pi . Also, when $(|x|; -y) \to +\infty$,the stresses vanish. Shearing module vary with depth is as :

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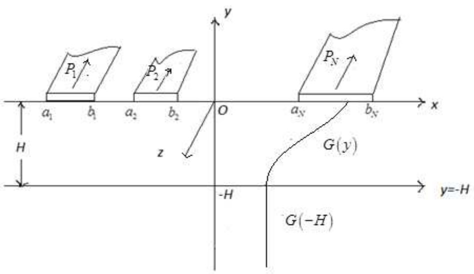

FIGURE 1. Schematic diagram of the contact problem for a half-space with an inhomogeneous coating.

$$i)G = G(y), -H \leq y \leq 0. \tag{2.1}$$

$$ii)G = G(-H), -\infty < y < -H.$$

Boundary conditions;

$$y = 0, \sigma_y = \tau_{xy} = 0,$$

$$\tau_{yz} = 0, x \notin \bigcup_{i=1}^{N}(a_i, b_i),$$

$$\omega_i^{(1)} = \varepsilon_i, x \in (a_i, b_i), i = \overline{1, N}, \tag{2.2}$$

where σ_y, τ_{xy} and τ_{yz} are normal and shear stress components, ω is the displacement along the axis z. The continuity conditions on layer substrate interface are given as

$$y = -H, \tau_{yz}^{(1)} = \tau_{yz}^{(2)}, \omega^{(1)} = \omega^{(2)}. \tag{2.3}$$

While $y = 0$,

$$\tau_{yz}|_{y=0} = \tau(x), \tau(x) = \begin{cases} \tau_i(x), & x \in \bigcup_{i=1}^{N}(a_i, b_i), \\ 0, & x \notin \bigcup_{i=1}^{N}(a_i, b_i), \end{cases} \tag{2.4}$$

$$\varepsilon = \begin{cases} \varepsilon_i, & x \in \bigcup_{i=1}^{N}(a_i, b_i), \\ 0, & x \notin \bigcup_{i=1}^{N}(a_i, b_i), \end{cases} \tag{2.5}$$

$$P_i = \int_{a_i}^{b_i} \tau_i(\xi) d\xi,$$

where, $i = \overline{1, N}$.

3 Reduction to a system of integral equation

In this section, the problem is reduced to a system of integral equation by using the basic equations of elasticity theory. Equilibrium equation is

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.$$

In the considered case, the following equations are satisfied:

$$u = v = 0, \omega = \omega(x, y), \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0.$$

Hooke's law is given as

$$\tau_{xz} = G(y)\frac{\partial \omega}{\partial x}, \tau_{yz} = G(y)\frac{\partial \omega}{\partial y}. \tag{3.1}$$

Conjugating displacements and stresses by Hooke's law leads to

$$G(y)\triangle\omega + G^{'}(y)\frac{\partial w}{\partial y} = 0,$$

where

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

From Eqs. (2.1)-(2.4),

$$G(y)\triangle\omega^{(1)} + G^{'}(y)\frac{\partial w^{(1)}}{\partial y} = 0, -H \leq y \leq 0, \tag{3.2}$$

$$\triangle\omega^{(2)} = 0, -\infty < y < -H. \tag{3.3}$$

Fourier transform of the displacement along the axis z is given as

$$\omega^{(j)}(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_j(\alpha, y)e^{-i\alpha x}d\alpha, j = 1, 2, \tag{3.4}$$

where the case of $j = 1$ correspond to nonhomogeneous and $j = 2$ corresponds to homogeneous half space. By inserting Eq. (3.4) into Eqs. (3.2)-(3.3) and considering the inverse Fourier transform the system of ordinary differential equations is obtained

$$G(y)W_1^{''} + G^{'}(y)W_1^{'} - \alpha^2 G(y)W_1 = 0, -H \leq y \leq 0 \tag{3.5}$$

$$W_2^{''} - \alpha^2 W_2 = 0, -\infty < y < -H.$$

The solution of the second equation of Eq. (3.5) is obtained as

$$W_2(\alpha, y) = B_1(\alpha)e^{|\alpha|y} + B_2(\alpha)e^{-|\alpha|y}.$$

Since the displacements should be finite when $y \to -\infty$, it should be that $B_2(\alpha) = 0$. So,

$$W_2(\alpha, y) = B_1(\alpha)e^{|\alpha|y}. \tag{3.6}$$

By considering the continiutiy condition (2.3),

$$W_1(\alpha,-H)=W_2(\alpha,-H), W_1^{'}(\alpha,-H)=W_2^{'}(\alpha,-H) \tag{3.7}$$

is obtained. Inserting Eq. (3.6) into (3.7) leads to

$$\frac{W_1^{'}(\alpha,-H)}{W_1(\alpha,-H)}=|\alpha|. \tag{3.8}$$

Expressing the function $\tau(x)$ by Fourier transform, it is obtained that

$$\tau_{yz}|_{y=0}=\tau(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}T(\alpha)e^{-i\alpha x}d\alpha, \tag{3.9}$$

where $T(\alpha)$ is the inverse Fourier transform of $\tau(x)$ defined by

$$T(\alpha)=\int_{-\infty}^{\infty}\tau(\xi)e^{i\alpha\xi}d\xi. \tag{3.10}$$

By using the initial condition (2.4) and Hooke's law (3.1) and also, considering the Fourier transform expressions (3.4) and (3.9) ,

$$W_1^{'}(\alpha,y)|_{y=0}=\frac{T(\alpha)}{G(0)} \tag{3.11}$$

is obtained. Now, let us define the following auxiliary function;

$$W_1^{*}(\alpha,y)=W_1(\alpha,y)G(0)|\alpha|T^{-1}(\alpha). \tag{3.12}$$

By rearranging Eqs. (3.8) and (3.11) by considering the auxilary function, the following equations are obtained:

$$\frac{W_1^{*'}(\alpha,-H)}{W_1^{*}(\alpha,-H)}=|\alpha|, \tag{3.13}$$

$$W_1^{*'}(\alpha,y)|_{y=0}=|\alpha|. \tag{3.14}$$

So, taking into consideration Eq. (3.12) and the first part of the Eq. (3.5), the following ordinary differential equation is obtained:

$$G(y)W_1^{*''}+G^{'}(y)W_1^{*'}-\alpha^2G(y)W_1^{*}=0. \tag{3.15}$$

By using the condition (2.2), the Fourier transform (3.4), the notation (2.4) and the auxilary function (3.12), the following equation can be written as

$$\frac{1}{2\pi G(0)}\int_{-\infty}^{\infty}\frac{W_1^{*}(\alpha,0)T(\alpha)}{|\alpha|}e^{-i\alpha x}d\alpha=\varepsilon.$$

To derive the integral equation, the function $W_1^{*}(\alpha,0)$ should be determined from Eq. (3.15) with the help of Eqs. (3.13) and (3.14). Here, instead of that, to benefit from the method presented in [9], it is assumed that

$$z_1=W_1^{*}(\alpha,y), z_2=W_1^{*'}(\alpha,y).$$

From the first part of the Eq. (3.5), for $G(y) \neq 0$, $\forall y \varepsilon (-H, 0)$ and with the help of Eqs. (3.13)-(3.14), the following system of differential equation is obtained:

$$\frac{d\overrightarrow{z}}{dy} = A\overrightarrow{z}, -H \leq y \leq 0, \tag{3.16}$$

$$\frac{z_2}{z_1}|_{y=-H} = |\alpha|, z_2|_{y=0} = |\alpha|,$$

where,

$$\overrightarrow{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ \alpha^2 & -\delta(y) \end{pmatrix}$$

and

$$\delta(y) = \frac{G^{'}(y)}{G(y)} \tag{3.17}$$

The solution of the system of differential equation is seek with the help of the methodology given in [9] as

$$\overrightarrow{z}(y) = \beta_1(\alpha)\overrightarrow{a}(\alpha, y)e^{|\alpha|y} \tag{3.18}$$

From Eqs. (3.16)-(3.18), it is obtained that

$$\frac{d\overrightarrow{a}}{dy} = A\overrightarrow{a} - |\alpha|\overrightarrow{a}, -H \leq y \leq 0, \tag{3.19}$$

$$a_1(\alpha, -H) = 1, a_2(\alpha, -H) = \alpha, \tag{3.20}$$

where a_1 and a_2 are the components of vector $\overrightarrow{a}$. So the function $W_1^*(\alpha, 0)$ is obtained:

$$W_1^*(\alpha, 0) = a_1(\alpha, 0)a_2^{-1}(\alpha, 0)|\alpha|.$$

So, Eq. (2.2), (2.4), (2.5) can be arranged as

$$\frac{1}{2\pi G(0)} \int_{-\infty}^{\infty} \frac{a_1(\alpha, 0)}{a_2(\alpha, 0)} T(\alpha)e^{-i\alpha x}d\alpha = \varepsilon. \tag{3.21}$$

4 A special case of shearing module

In this section, as a special case, the following conditions will be taking into consideration to constract the kernels of integral equations:

$$i) G(y) = Ge^{\nu y}, -H \leq y \leq 0,$$
$$ii) G(y) = Ge^{-\nu y}, -\infty < y < -H.$$

where ν is the nonhomogeneity parameter controlling the variation of the shear modulus in the coating medium.

It is necessary to determine $a_1(\alpha, 0)$ and $a_2(\alpha, 0)$ to derive the integral equation under these conditions. To do that, by solving the system of differential equations (3.19)-(3.20) and considering Eq. (3.17), Eq. (3.21) can be expressed as

$$\frac{1}{2\pi G(0)} \int_{-\infty}^{\infty} \frac{(\frac{\nu}{2} + \theta + |\alpha|)e^{2\theta H} - \frac{\nu}{2} + \theta - |\alpha|}{((-\frac{\nu}{2} + \theta)|\alpha| + \alpha^2)e^{2\theta H} + (\frac{\nu}{2} + \theta)|\alpha| - \alpha^2} T(\alpha)e^{-i\alpha x}d\alpha = \varepsilon.$$

By using the inverse transform of defined by Eq. (3.10),

$$\int_{-\infty}^{\infty} \tau(\xi)(\int_{-\infty}^{\infty} \frac{L(u)}{|u|} e^{-iu(\frac{x-\xi}{H})} du) d\xi = 2\pi G(0)\varepsilon,$$

where,

$$L(u) = \frac{V \cosh V + (|u| + \mu) \sinh V}{V \cosh V + (|u| - \mu) \sinh V}, \alpha H = u, \mu = \frac{\nu H}{2}, V = \sqrt{\mu^2 + u^2}.$$

Since the function $L(u)$ is even with respect to u, the following system of integral equation is achieved:

$$\int_{-\infty}^{\infty} \tau(\xi) K(\frac{\xi - x}{H}) d\xi = \pi G(0)\varepsilon, \tag{4.1}$$

where,

$$K(t) = \int_{0}^{\infty} \frac{L(u)}{u} cos(ut) du.$$

The kernel has the following properties:

Theorem 4.1. The function $K(t)$ defined by

$$K(t) = \int_{0}^{\infty} \frac{L(u)}{u} cos(ut) du.$$

can be rewritten for $\forall t \varepsilon (-\infty, \infty)$ as

$$K(t) = -\ln|t| - F(t),$$

where,

$$F(t) = \int_{0}^{\infty} \frac{(1 - L(u)) cos(ut) - e^{-u}}{u} du$$

[10].

The Eqs. (4.1) can be rewritten by using the equations given in the theorem and (2.5):

$$-\sum_{j=1}^{N} \int_{a_j}^{b_j} \tau_j(\xi)[\ln(\frac{\xi - x}{H}) + F(\frac{\xi - x}{H})] d\xi = \pi G(0)\varepsilon_i, x\varepsilon(a_i, b_i), i = \overline{1, N}$$

By derivating the system above with respect to x and considering Eq. (4.2)

$$\sum_{j=1}^{N} \int_{a_j}^{b_j} \frac{\tau_j(\xi)}{\xi - x} d\xi + \frac{1}{H} \sum_{j=1}^{N} \int_{a_j}^{b_j} \tau_j(\xi) M(\xi, x) d\xi = \pi G(0)\varepsilon_i^{'}, x\varepsilon[a_i, b_i], i = \overline{1, N}$$

is obtained, where

$$M(\xi, x) = \int_{0}^{\infty} (1 - L(u)) sin(\frac{\xi - x}{H} u) du. \tag{4.2}$$

Here, only the integral in the first sum of the case of $j = i$ is the Cauchy type singular integral equation. The other integrals are regular. Let the notations be

$$\xi = \eta_j(\gamma) = r_j\gamma + s_j, x = \eta_i(t) = r_i t + s_i, \tag{4.3}$$

where,

$$r_j = \frac{b_j - a_j}{2}, s_j = \frac{b_j + a_j}{2}.$$

So, while $\xi\varepsilon[a_j, b_j]$; then $\gamma\varepsilon[-1,1]$. Similarly, while $x = \eta_i(t) = r_i t + s_i\varepsilon[a_i, b_i]$; then $t\varepsilon[-1,1]$. So the obtained system can be written in the dimensionless form as

$$\sum_{j=1}^{N}\int_{-1}^{1}\frac{\tau_j(\eta_j(\gamma))}{\eta_j(\gamma)-\eta_i(t)}\eta_j'(\gamma)d\gamma+\frac{1}{H}\sum_{j=1}^{N}\int_{-1}^{1}\tau_j(\eta_j(\gamma))M(\eta_j(\gamma),\eta_i(t))\eta_j'(\gamma)d\gamma = \pi G(0)\varepsilon_i', t\varepsilon[-1,1], i=\overline{1,N}$$

$$\int_{-1}^{1}\frac{\tau_i(\eta_i(\gamma))}{\eta_i(\gamma)-\eta_i(t)}\eta_i'(\gamma)d\gamma+\sum_{\substack{j=1\\ j\neq i}}^{N}\int_{-1}^{1}\frac{\tau_j(\eta_j(\gamma))}{\eta_j(\gamma)-\eta_i(t)}\eta_j'(\gamma)d\gamma+\frac{1}{H}\sum_{j=1}^{N}\int_{-1}^{1}\tau_j(\eta_j(\gamma))M(\eta_j(\gamma),\eta_i(t))\eta_j'(\gamma)d\gamma = \pi G(0)\varepsilon_i'$$

Assuming,

$$\lambda_j = \frac{H}{\eta_j'(\gamma)} = \frac{H}{r_j}, \varphi_j(\gamma) = \frac{1}{G(0)}\tau_j(\eta_j(\gamma)) \;\; and \;\; \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4.4}$$

leads to

$$\int_{-1}^{1}\frac{\varphi_i(\gamma)}{\gamma - t}d\gamma + \sum_{\substack{j=1\\ j\neq i}}^{N}\int_{-1}^{1}\frac{\varphi_j(\gamma)}{\gamma - \frac{r_i}{r_j}t + \frac{s_j-s_i}{r_j}}d\gamma + \sum_{j=1}^{N}\frac{1}{\lambda_j}\int_{-1}^{1}\varphi_j(\gamma)M\left(\frac{\gamma - \frac{r_i}{r_j}t + \frac{s_j-s_i}{r_j}}{\lambda_j}\right)d\gamma = \pi\varepsilon_i'. \tag{4.5}$$

Let the function $K_{ij}^*(\gamma, t)$ be

$$K_{ij}^*(\gamma,t) = \frac{1}{\pi}\left[\frac{1-\delta_{ij}}{\gamma - \frac{r_i}{r_j}t + \frac{s_j-s_i}{r_j}} + \frac{1}{\lambda_j}M\left(\frac{\gamma - \frac{r_i}{r_j}t + \frac{s_j-s_i}{r_j}}{\lambda_j}\right)\right]. \tag{4.6}$$

By rearranging, the following singular integral equation system is obtained

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi_i(\gamma)}{\gamma - t}d\gamma + \sum_{j=1}^{N}\int_{-1}^{1}K_{ij}^*(\gamma,t)\varphi_j(\gamma)d\gamma = \varepsilon_i', i = \overline{1,N}. \tag{4.7}$$

5 A quadrature approach to solution of the system of singular integral equation

In this section, the solution of the system of singular integral equations (4.7) will be approached by a quadrature approach given in [8]. It is assumed that the solution has the integrable singularities at the both ends ± 1 . According to index theory given in [7], since $\kappa = 1$,the solution is of the form

$$\varphi_j(t) = w(t)\varphi_j(t), w(t) = \frac{1}{\sqrt{1-t^2}}, j = \overline{1,N}, \tag{5.1}$$

where $\varphi_j(t)$ is the bounded function, $w(t)$ is the weight function. Inserting Eq. (5.1) into the system of singular integral equations (4.7), the following system of singular integral equation is obtained:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi_i(\gamma)}{(\gamma-t)\sqrt{1-\gamma^2}}d\gamma+\sum_{j=1}^{N}\int_{-1}^{1}\frac{\varphi_j(\gamma)}{\sqrt{1-\gamma^2}}K_{ij}^{*}(\gamma,t)d\gamma=\varepsilon_i^{'}. \tag{5.2}$$

Let us express the function $\varphi_i(\gamma)$ by the truncated series as

$$\varphi_i(\gamma)=\sum_{m=0}^{p}B_{im}T_m(\gamma), \tag{5.3}$$

where $T_m(\gamma)$ are the first kind Chebyshev polynomials of order m. So, in the view of the known relation [11],

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_m(\gamma)}{(\gamma-t)\sqrt{1-\gamma^2}}d\gamma=U_{m-1}(t), m>0, -1<t<1,$$

$$\sum_{k=1}^{n}\frac{T_j(\gamma_k)}{n(\gamma_k-tr)}=\begin{cases}0, & j=0,\\ U_{j-1}(t_r), & 0<j<n,\end{cases} \tag{5.4}$$

where, $U_{m-1}(t)$ are the second kind Chebyshev polynomials of order $m-1$ and

$$T_n(\gamma_k)=0, \gamma_k=cos(\frac{\pi(2k-1)}{2n}), k=1,...,n,$$

$$U_{n-1}(t_r)=0, t_r=cos(\frac{\pi r}{n}), r=1,...,n-1,$$

the singular integral in Eq. (5.2) can be rewritten as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi_i(\gamma)}{(\gamma-t)\sqrt{1-\gamma^2}}d\gamma=\sum_{m=0}^{p}B_{im}\frac{1}{\pi}\int_{-1}^{1}\frac{T_m(\gamma)}{(\gamma-t)\sqrt{1-\gamma^2}}d\gamma=\sum_{m=1}^{p}B_{im}U_{m-1}(t).$$

For $t=t_r$, using the relation (5.3) and (5.4), it is obtained that

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi_i(\gamma)}{(\gamma-t_r)\sqrt{1-\gamma^2}}d\gamma=\sum_{m=1}^{p}B_{im}U_{m-1}(t_r)=\sum_{k=1}^{n}\frac{\sum_{m=1}^{p}B_{im}T_m(\gamma_k)}{n(\gamma_k-t_r)}=\sum_{k=1}^{n}\frac{\varphi_i(\gamma_k)}{n(\gamma_k-tr)}.$$

So, under the consideration of the case of flat base punches, the system of singular integral equations (4.5) turns into

$$\sum_{k=1}^{n}\frac{\varphi_i(\gamma_k)}{n(\gamma_k-tr)}+\sum_{j=1}^{N}\sum_{k=1}^{n}\frac{\pi}{n}\varphi_j(\gamma_k)K_{ij}^{*}(\gamma_k,t_r)=0, \tag{5.5}$$

where, the second sum of the system (5.5) is obtained by applying the known Gauss-Chebyshev polynomial given as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{h(t)}{\sqrt{1-t^2}}dt=\sum_{k=1}^{n}\frac{h(t_k)}{n}, T_n(t_k)=0$$

to the regular part of Eq. (5.2). As is seen from Eq. (5.5),while the number of the equations is $N(n-1)$, the number of unknown is Nn. So the number of missing equations is N. By the index theory, since $\kappa = 1$, there should be additional condition. So as the additional conditions, since there are N puches the following static conditions are written

$$\int_{a_j}^{b_j} \tau_j(\xi)d\xi = P_j, j = \overline{1, N}. \tag{5.6}$$

By considering the notations (4.3) and transform (4.4), the additional condition (5.6) turn into

$$\int_{-1}^{1} \varphi_j(\gamma)d\gamma = P_j^*, j = \overline{1, N}, \tag{5.7}$$

where,

$$P_j^* = \frac{P_j}{r_j G(0)}.$$

By applying the same procedure to Eq. (5.7),

$$\sum_{k=1}^{n} \frac{\pi}{n}\varphi_j(\gamma_k) = P_j^*, j = \overline{1, N} \tag{5.8}$$

is obtained.So, the solution of the considered problem is reduced to find the unknown functions $\varphi_j(\gamma_k)$ from the linear algebraic systems (5.5) and (5.8).

6 Numerical illustrations

In this section the detailed numerical illustrations are given for the different cases of the punch number N. It is assumed that every punch is loaded by same forces. In tables, while t_1^*, t_2^*, t_3^* denotes the minimum values of the functions τ_1, τ_2 and τ_3 respectively; M_1, M_2 and M_3 denotes the momentum of the punches.

6.1 Case of $N = 1$

In this case, it is assumed that there is only one punch which is in contact with the elastic inhomogeneous coating. Numerical results are presented below.

TABLE 1. Numerical results for different values of nonhomogenity parameter ν on the interval $[-1, 1]$.

ν	t^*	$\tau(t^*)$	M
0	0	0.318309	0
0.45	0	0.34683	0
1	0	0.407306	0

6.2 Case of $N = 2$

In this case, it is assumed that there are two punches which are in contact with the elastic inhomogeneous coating. It is assumed that the problem is symmetric according to axis y. Numerical results are presented below.

TABLE 2. Numerical results for different values of nonhomogenity parameter ν on the interval $[-2,-1]\cup[1,2]$.

ν	t_1^*	$\tau_1(t_1^*)$	t_2^*	$\tau_2(t_2^*)$	M_1	M_2
0	-1.41423	0.63662	1.41423	0.63662	-1.54196	1.54196
0.45	-1.44066	0.67319	1.44066	0.67319	-1.52456	1.52456
1	-1.55831	0.72710	1.58831	0.72710	-1.48441	1.48441

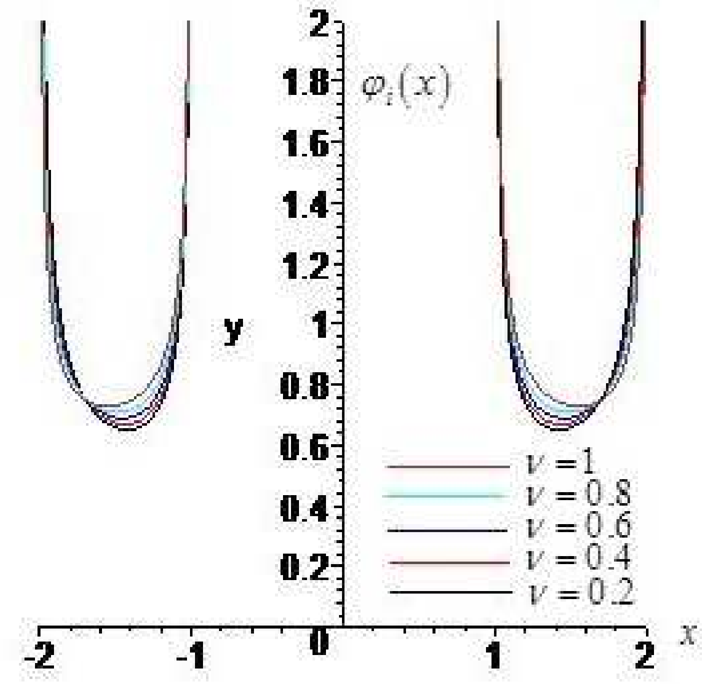

FIGURE 2. $\varphi_i(x)$ for the different values of nonhomogeneity parameter.

6.3 Case of N=3

In this case, it is assumed that there are three punches which are in contact with the elastic inhomogeneous coating. Numerical results are presented below.

TABLE 3. Numerical results for different values of nonhomogenity parameter ν on the interval $[-5,-3]\cup[-1,1]\cup[3,5]$.

ν	t_1^*	$\tau_1(t_1^*)$	t_2^*	$\tau_2(t_2^*)$	t_3^*	$\tau_3(t_3^*)$	M_1	M_2	M_3
0	-3.57362	0.30558	0	0.33842	3.57362	0.30558	-4.19311	0	4.19311
0.45	-3.67993	0.34322	0	0.36101	3.67993	0.34322	-4.11191	0	4.11191
1	-4.66281	0.34592	0	0.40867	4.66281	0.34592	-3.85018	0	3.85018

TABLE 4. Numerical results for different intervals and nonhomogeneity parameters.

ν	$[a_1,b_1]$	$[a_2,b_2]$	$[a_3,b_3]$	t_1^*	$\tau_1(t_1^*)$	t_2^*	$\tau_2(t_2^*)$	t_3^*	$\tau_3(t_3^*)$	M_1	M_2	M_3
0	[-5,-3]	[-1,1]	[3,5]	-3.573	0.305	0	0.338	3.573	0.305	-4.193	0	4.193
	[-6,-4]	[-1,1]	[4,6]	-4.676	0.310	0	0.331	4.676	0.310	-5.152	0	5.152
	[-7,-5]	[-1,1]	[5,7]	-5.736	0.313	0	0.327	5.736	0.313	-6.126	0	6.126
	[-8,-6]	[-1,1]	[6,8]	-6.777	0.314	0	0.324	6.777	0.314	-7.108	0	7.108
0.45	[-5,-3]	[-1,1]	[3,5]	-3.679	0.343	0	0.361	3.679	0.343	-4.111	0	4.111
	[-6,-4]	[-1,1]	[4,6]	-4.760	0.344	0	0.355	4.760	0.344	-5.085	0	5.085
	[-7,-5]	[-1,1]	[5,7]	-5.807	0.345	0	0.352	5.807	0.345	-5.085	0	5.085
	[-8,-6]	[-1,1]	[6,8]	-6.838	0.345	0	0.350	6.838	0.345	-7.057	0	7.057
1	[-5,-3]	[-1,1]	[3,5]	-4.662	0.345	0	0.408	4.662	0.345	-3.850	0	3.850
	[-6,-4]	[-1,1]	[4,6]	-5.634	0.354	0	0.404	5.634	0.354	-4.862	0	4.862
	[-7,-5]	[-1,1]	[5,7]	-6.605	0.362	0	0.403	6.605	0.362	-5.876	0	5.876
	[-8,-6]	[-1,1]	[6,8]	-7.578	0.369	0	0.403	7.578	0.369	-6.889	0	6.889

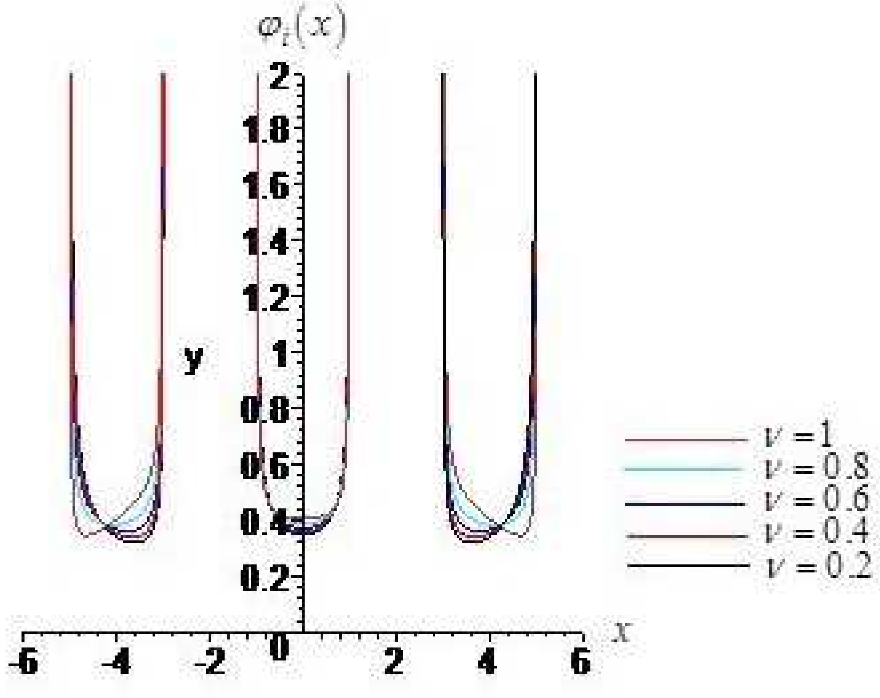

FIGURE 3. $\varphi_i(x)$ for the different values of nonhomogeneity parameter.

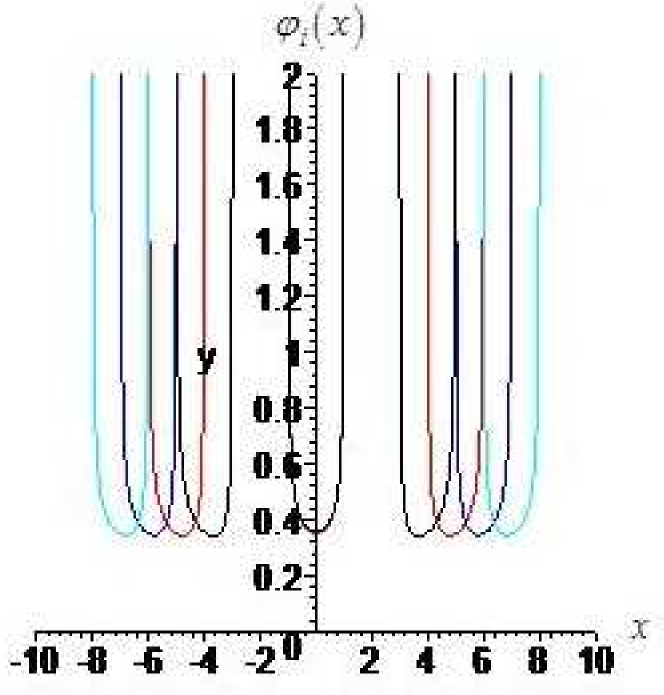

FIGURE 4. $\varphi_i(x)$ for $\nu = 1$ on the different intervals.

7 Conclusion

In this study, the generalization of contact problem in an inhomogeneous half space is studied. The problem is modelled as system of singular integral equations. So, the obtained system is solved with the help of Chebyshev polynomials. The numerical results are given in tabular and graphical form for the cases of $N = 1, N = 2, N = 3$. The effect of the nonhomogeneity parameter on pressure distribution is examined. Also, pressure distribution is presented for the different contact intervals.On the basis of tables and figures, the results can be interpreted as follows:

For the case of $N = 1$: While the nonhomogeneity parameter increases, the minimum pressure distribution increases, too. The minimum value t^* is equal to zero, since the interval is chosen symmetrically. Also, the momentum of the punch is equal to zero due to the same reason.

For the case of $N = 2$: When the nonhomogeneity parameter increases, the minimum pressure distribution on punches increases. The minimum values t_1^* and t_2^* are symmetric, since the interval is chosen symmetrically. Also, due to the same reason, the momentum values are symmetric.

For the case of $N = 3$: When the nonhomogeneity parameter increases, the minimum pressure distribution increases. The minimum values t_1^*, t_3^* and the momentums M_1 and M_3 are symmetric. The minimum value t_2^* and also the momentum M_2 are equal to zero, since the interval is chosen symmetrically. When the punch 1 and punch 3 go far from the punch 2 preserving the symmetry and the interval width, the effect of the punch 1 and punch 3 on punch 2 decreases, so the minimum pressure distribution $\tau_2(t_2^*)$ decreases, too.

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